Math 4550 - Homework # 4 - Homomorphisms

Part 1 - Computations

1. Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be given by $\phi(x) = 3x$.
   (a) Draw a picture of ϕ.
   (b) Find the kernel of ϕ and the image of ϕ and add them to your picture.
   (c) Prove that ϕ is a homomorphism.
   (d) Prove that ϕ is one-to-one but not onto.

2. Let $\phi : \mathbb{R}^* \to \mathbb{R}^*$ be given by $\phi(x) = x^3$.
   Recall that $\mathbb{R}^*$ is a group under multiplication.
   (a) Draw a picture of ϕ.
   (b) Find the kernel of ϕ and the image of ϕ and add them to your picture.
   (c) Prove that ϕ is an isomorphism.

3. Let $\phi : GL(2, \mathbb{R}) \to \mathbb{R}^*$ be given by $\phi(A) = \det(A)$.
   (a) Draw a picture of ϕ.
   (b) Prove that ϕ is a homomorphism.
   (c) Prove that ϕ is onto, but that it is not one-to-one.
   (d) Find the kernel of ϕ and the image of ϕ.

4. Suppose that $\phi : \mathbb{Z} \to \mathbb{Z}$ is a homomorphism such that $\phi(1) = 5$.
   (a) Find $\phi(4)$.
   (b) Find $\phi(-1)$.
   (c) Find $\phi(-3)$.

Part 2 - Proofs

5. Let G_1 and G_2 be groups and $\phi : G_1 \to G_2$ be a homomorphism.
   (a) Prove that if G_1 is abelian and ϕ is onto, then G_2 is abelian.
   (b) Prove that $\mathbb{Z}_6$ is not isomorphic to D_6.
   (c) Prove that if G_1 is cyclic and ϕ is onto, then G_2 is cyclic.
   (d) Prove that $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$

6. Let G_1 and G_2 be groups with identity elements e_1 and e_2 respectively. Let $\phi : G_1 \to G_2$ be a homomorphism.

   (a) Show that $\phi(e_1) = e_2$.

   (b) Show that $\phi(x^{-1}) = [\phi(x)]^{-1}$ for all $x \in G_1$.

   (c) Show that $\ker(\phi)$ is a subgroup of G_1.

   (d) Show that $\text{im}(\phi) = \{\phi(x) \mid x \in G\}$ is a subgroup of G_2.

   (e) Show that ϕ is one-to-one if and only if $\ker(\phi) = \{e_1\}$.

   (f) Show that ϕ is onto if and only if $\text{im}(\phi) = G_2$.

7. Let G_1, G_2, H_1, H_2 be groups. Prove that if $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1 \times H_1 \cong G_2 \times H_2$.
